# Weighted first-order logic, weighted aperiodic automata, and more

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Manfred Droste & P.G., Aperiodic Weighted Automata and Weighted First-Order Logic, MFCS'19
+ ongoing work

# Words — ~~Boolean~~ Weighted

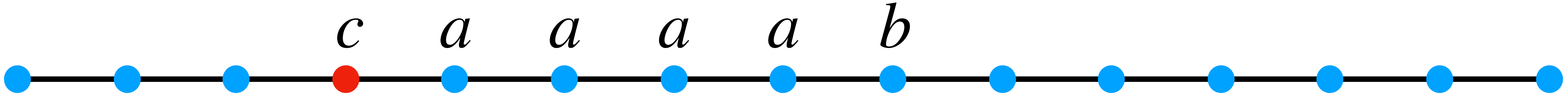
TFAE:

1. First-order logic (FO)
2. Linear Temporal Logic (LTL)
3. Aperiodic Languages (or Automata) (AP)
4. Star-Free Languages (SF)
5. Star-Free Propositional Dynamic Logic (SFPDL)

FO=LTL Kamp PhD 1968

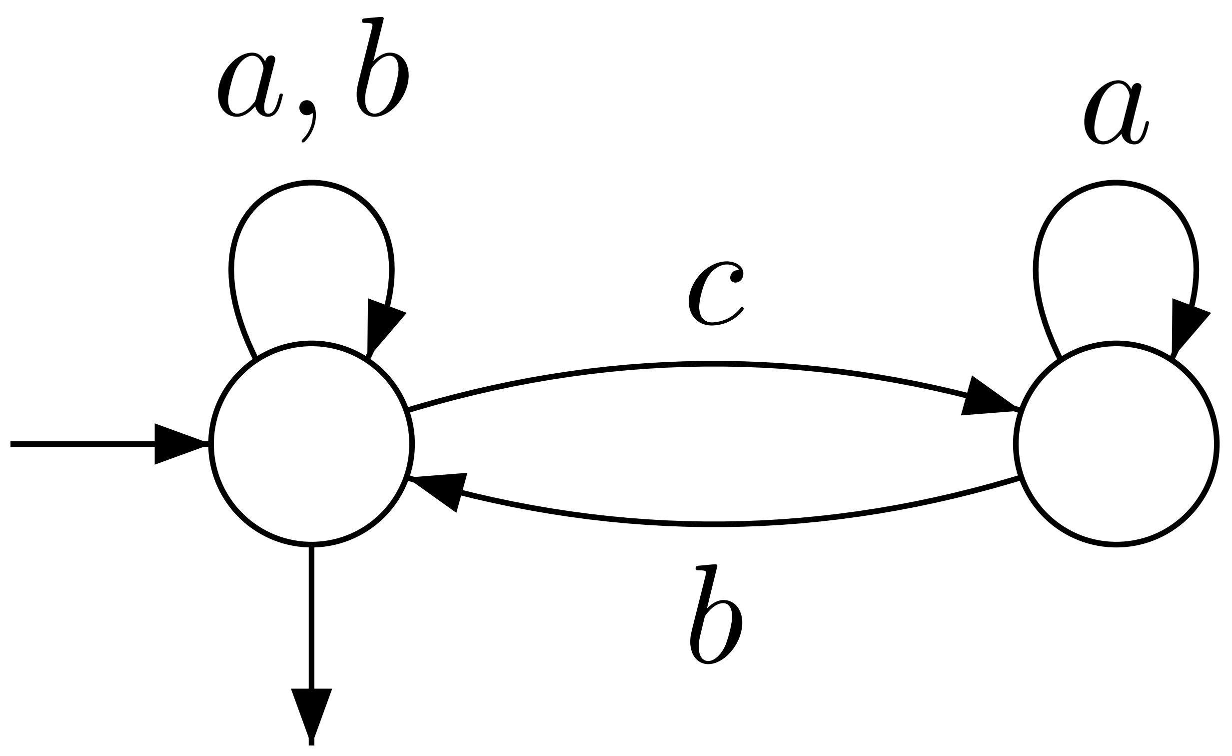
AP=SF Schützenberger I&C 1965

SF=FO McNaughton & Papert 1971

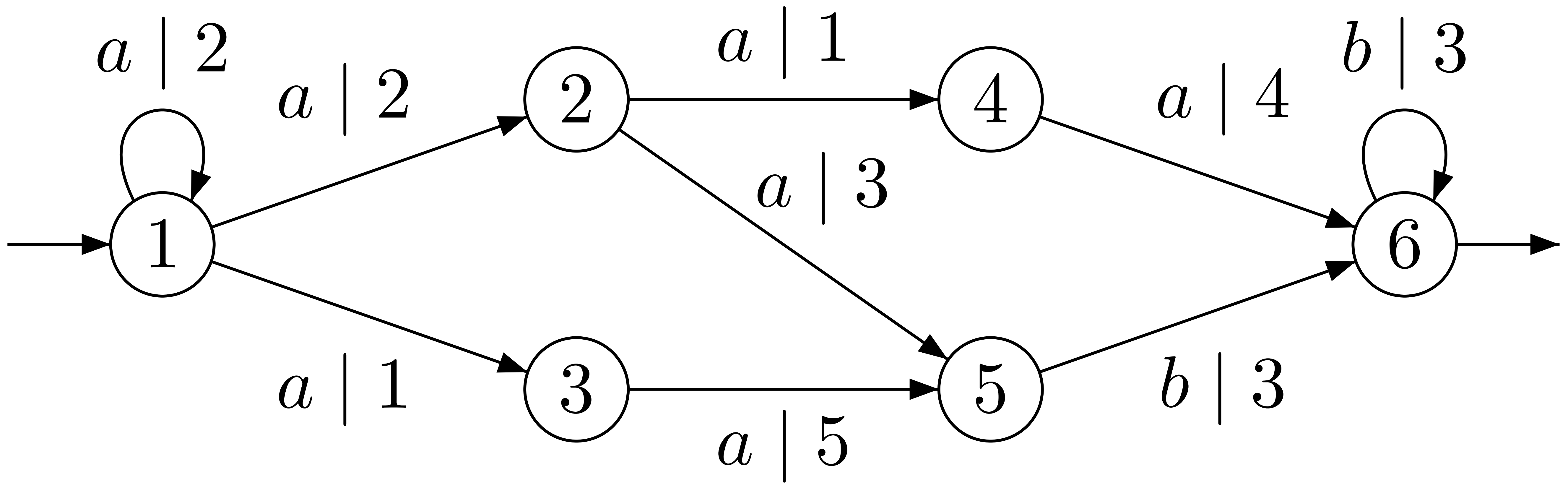



$$\forall x \left( c(x) \to \exists z \left( z > x \wedge b(z) \wedge \forall y \left( x < y < z \to a(y) \right) \right) \right)$$

$$\mathsf{G}\left( c \to \mathsf{X}(a \,\mathsf{U}\, b) \right)$$



# Weighted Automata (wA)




Input word: $w = a^n a^3 b b^p$

- $1 \xrightarrow{a} 1 \ \cdots \ 1 \xrightarrow{a} 2 \xrightarrow{a} 4 \xrightarrow{a} 6 \xrightarrow{b} 6 \ \cdots \ 6 \xrightarrow{b} 6$
- $1 \xrightarrow{a} 1 \ \cdots \ 1 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 5 \xrightarrow{b} 6 \ \cdots \ 6 \xrightarrow{b} 6$
- $1 \xrightarrow{a} 1 \ \cdots \ 1 \xrightarrow{a} 1 \xrightarrow{a} 3 \xrightarrow{a} 5 \xrightarrow{b} 6 \ \cdots \ 6 \xrightarrow{b} 6$

Weights of runs:

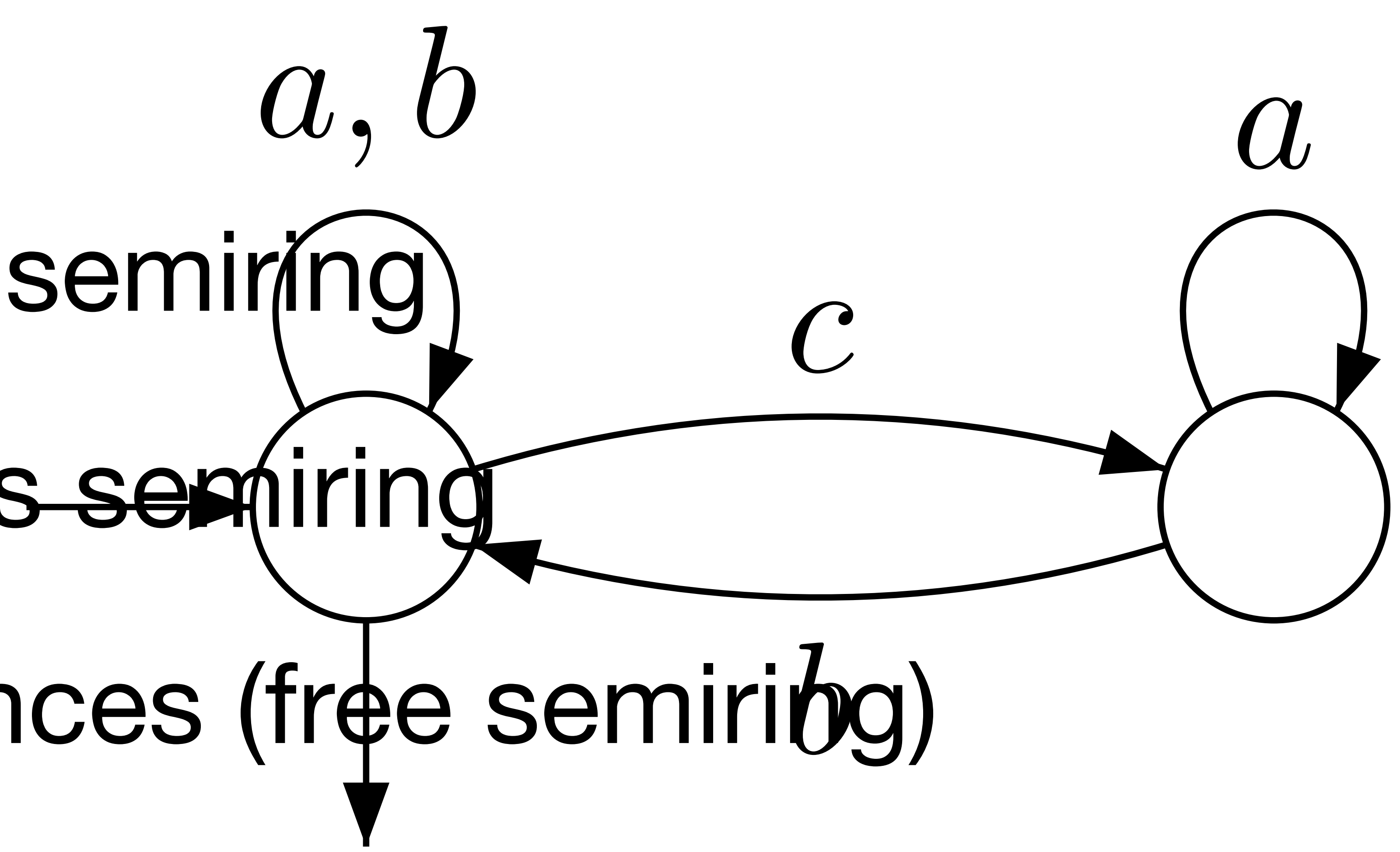
- $2^n \times 2 \times 1 \times 4 \times 3 \times 3^p$
- $2^n \times 2 \times 2 \times 3 \times 3 \times 3^p$
- $2^n \times 2 \times 1 \times 5 \times 3 \times 3^p$

$$[\![\mathscr{A}]\!](w) = \sum_\rho \mathrm{wt}(\rho) = 2^n \times (24 + 36 + 30) \times 3^p$$

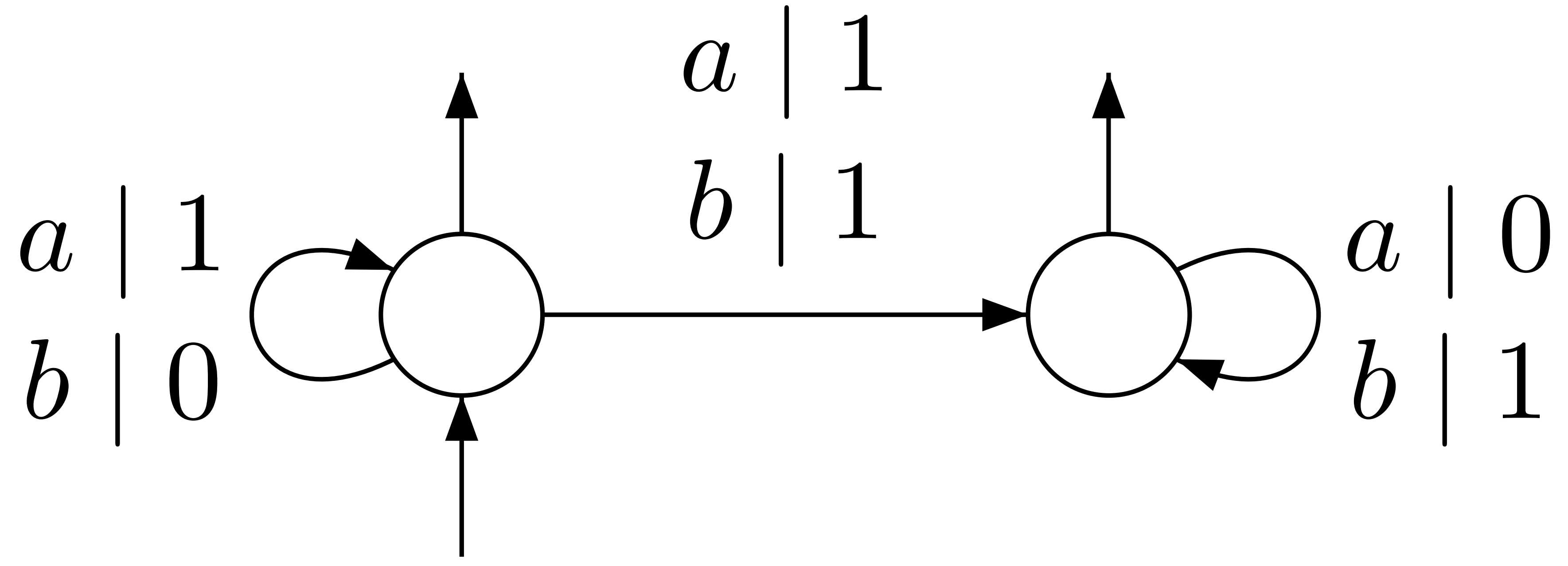
# Semirings

Examples:

1. $\mathbb{N}_{+,\times} = (\mathbb{N}, +, \times, 0, 1)$ : Natural numbers (usual addition and multiplication)
2. $\mathbb{B} = (\{\perp, \top\}, \vee, \wedge, \perp, \top)$ : Boolean semiring
3. $\mathbb{N}_{\min,+} = (\mathbb{N} \cup \{+\infty\}, \min, +, +\infty, 0)$ : min-plus semiring
4. $\mathbb{N}_{\max,+} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ : max-plus semiring
5. $(\mathbb{N}\langle \mathsf{R}^* \rangle, \uplus, \cdot, \emptyset, \{\{\varepsilon\}\})$ : multisets of weight sequences (free semiring)



# Example with max-plus




$\mathbb{N}_{\max,+} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ : max-plus semiring

$$[\![\mathscr{A}]\!](w) = \max\{\,|u|_a + |v|_b \mid w = uv\}$$

# Weighted First-Order Logic (wFO)

FO $\varphi ::= \top \mid a(x) \mid x \leq y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x \varphi$

step-wFO $\alpha ::= r \mid \varphi \,?\, \alpha : \alpha$

wFO $\Phi ::= \mathbf{0} \mid \prod_z \alpha \mid \sum_x \Phi \mid \Phi + \Phi \mid \varphi \,?\, \Phi : \Phi$

Example: $L = (ab + aab + c)^+$ is FO-definable

- $\varphi_L = \forall x \left(b(x) \to a(x-1)\right) \wedge \left(a(x) \to b(x+1) \vee (a(x+1) \wedge b(x+2))\right)$
- $\alpha(z) = \left(a(z) \vee c(z)\right) ? 1 : \left(a(z-2) ? 3 : 2\right)$
- $\Phi = \varphi_L \,?\, \prod_z \alpha(z) : \mathbf{0}$
- For $w \in L$, let $n = |w|_{ab}$ and $m = |w|_{aab}$. Then, $[\![\Phi]\!]^w = 3^m \times 2^{n-m}$

# Weighted First-Order Logic (wFO)

FO $\varphi ::= \top \mid a(x) \mid x \leq y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x \varphi$

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wFO $\Phi ::= \mathbf{0} \mid \prod_z \alpha \mid \sum_x \Phi \mid \Phi + \Phi \mid \varphi\,?\,\Phi : \Phi$

Semantics: $w \in \Sigma^+$ and $\nu : V \to \mathrm{pos}(w)$

- $[\![r]\!]^w_\nu = r$ and $[\![\varphi\,?\,\alpha_1 : \alpha_2]\!]^w_\nu = \begin{cases} [\![\alpha_1]\!]^w_\nu & \text{if } w, \nu \models \varphi \\ [\![\alpha_2]\!]^w_\nu & \text{otherwise.} \end{cases}$
- $[\![\mathbf{0}]\!]^w_\nu = 0$ and $[\![\prod_z \alpha]\!]^w_\nu = \prod_{j=1}^{|w|} [\![\alpha]\!]^w_{\nu[z\mapsto j]}$ and $[\![\sum_x \Phi]\!]^w_\nu = \sum_{j=1}^{|w|} [\![\Phi]\!]^w_{\nu[x\mapsto j]}$
- $[\![\Phi_1 + \Phi_2]\!]^w_\nu = [\![\Phi_1]\!]^w_\nu + [\![\Phi_2]\!]^w_\nu$ and $[\![\varphi\,?\,\Phi_1 : \Phi_2]\!]^w_\nu = \begin{cases} [\![\Phi_1]\!]^w_\nu & \text{if } w, \nu \models \varphi \\ [\![\Phi_2]\!]^w_\nu & \text{otherwise.} \end{cases}$

# Weighted First-Order Logic (wFO)

| | |
|---|---|
| FO | $\varphi ::= \top \mid a(x) \mid x \leq y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x \varphi$ |
| step-wFO | $\alpha ::= r \mid \varphi \,?\, \alpha : \alpha$ |
| wFO | $\Phi ::= \mathbf{0} \mid \prod_z \alpha \mid \sum_x \Phi \mid \Phi + \Phi \mid \varphi \,?\, \Phi : \Phi$ |

Example: Let $L \subseteq \Sigma^+$ be a language defined by the FO sentence $\varphi$.

$$\Phi_{\text{pref}} = \sum_x \varphi^{\leq x} \,?\, \mathbf{1} : \mathbf{0}$$

Number of prefixes in $L$

where $\mathbf{1} = \prod_z 1$ and $\varphi^{\leq x}$ is $\varphi$ relativized to the prefix $[1,x]$

# Weighted First-Order Logic (wFO)

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Example: Let $L \subseteq \Sigma^+$ be a language defined by the FO sentence $\varphi$.

$$\Phi_{\text{inf}} = \sum_{x,y} (x \leq y \wedge \varphi^{[x,y]}) \,?\, \mathbf{1} : \mathbf{0}$$

Number of factors in $L$

where $\mathbf{1} = \prod_z 1$ and $\varphi^{[x,y]}$ is $\varphi$ relativized to the infix $[x, y]$

# wFO vs wA

Theorem [Manfred Droste & P.G., 2019]

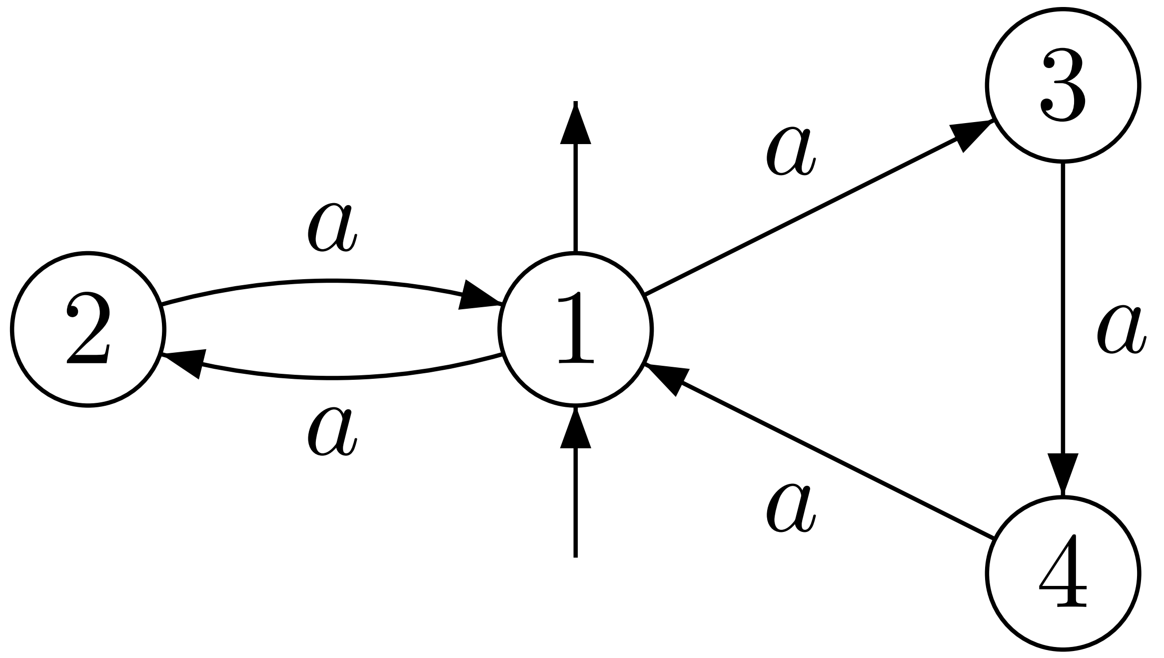
The following classes are expressively equivalent:

- Aperiodic polynomially ambiguous wA and wFO sentences,
- Aperiodic finitely ambiguous wA and wFO sentences without first-order sums $\left(\sum_x\right)$,
- Aperiodic unambiguous wA and wFO sentences without binary sums (+) or first-order sums $\left(\sum_x\right)$.

# Aperiodic automata

The transition monoid of $\mathscr{A}$ is aperiodic:

$$\exists n \geq 1, \forall u \in \Sigma^+, \forall p, q \in Q, \qquad p \xrightarrow{u^n} q \iff p \xrightarrow{u^{n+1}} q$$




$$3 \xrightarrow{a^1} 4 \qquad 3 \xrightarrow{a^6} 4$$

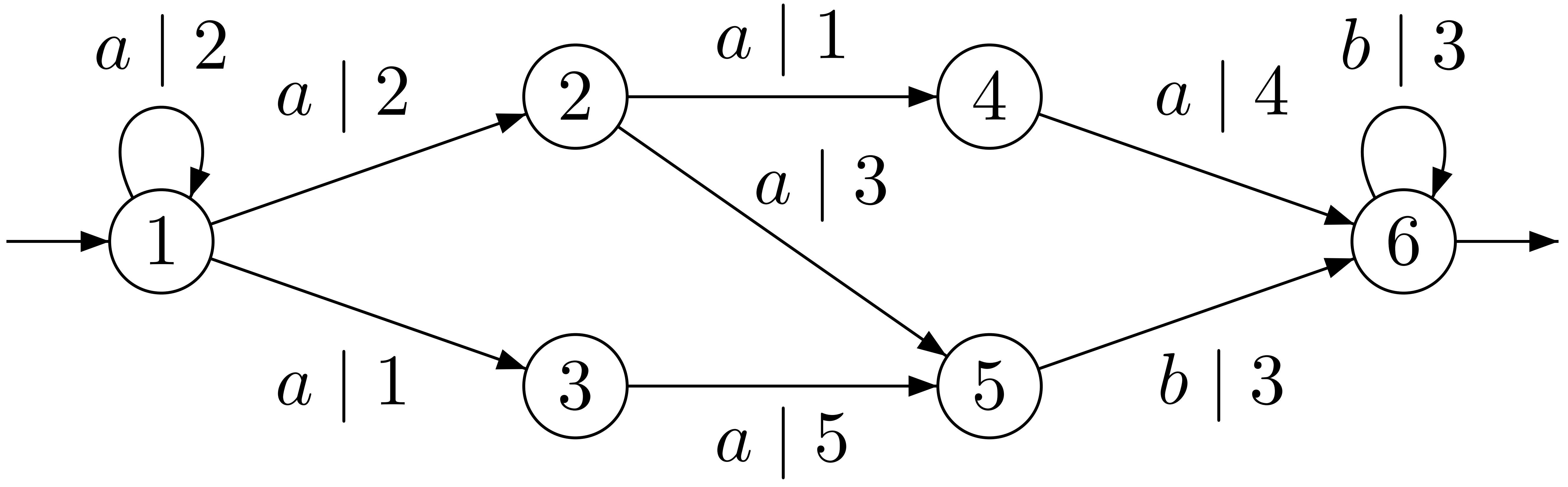
$$3 \xrightarrow{a^4} 4 \qquad 3 \xrightarrow{a^7} 4 \qquad n = 6$$

Theorem (Boolean) [Schützenberger 65, McNaughton, Papert 71]

Aperiodic automata = FO sentences

# Ambiguity in automata

$k$-ambiguous: Every word has at most $k$ accepting run.




Domain: $a^*(a^3 + a^2b)b^*$

3-ambiguous

Unambiguous = 1-ambiguous

Input word: $w = a^n a^3 b b^p$

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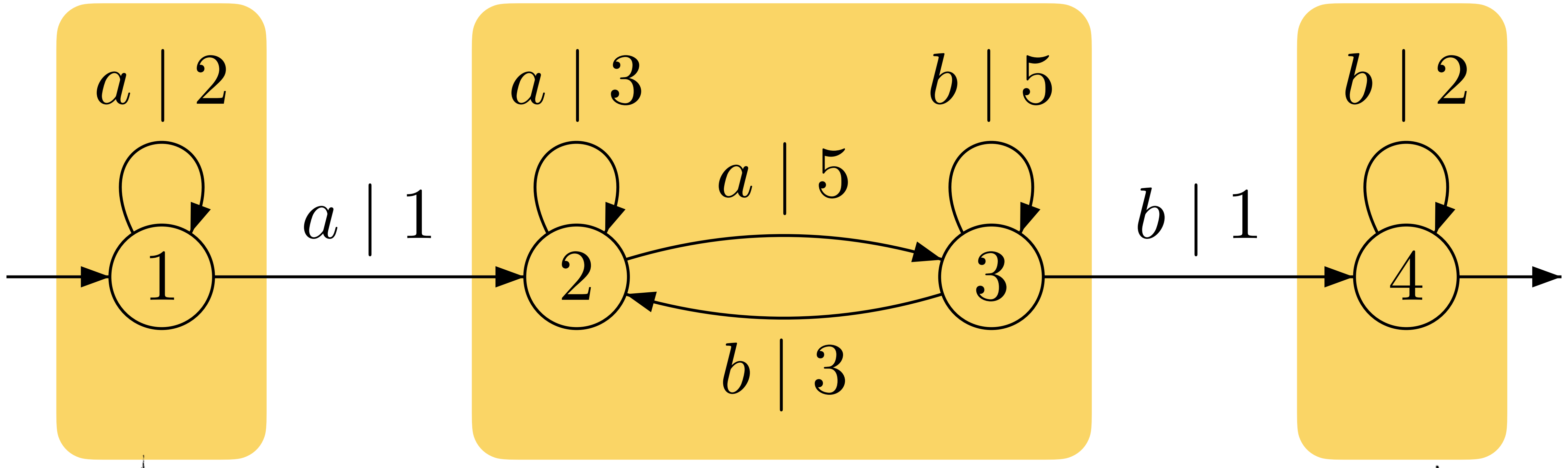
- $1 \xrightarrow{a} 1 \;\cdots\; 1 \xrightarrow{a} 1 \xrightarrow{a} 3 \xrightarrow{a} 5 \xrightarrow{b} 6 \;\cdots\; 6 \xrightarrow{b} 6$

# Ambiguity in automata

**Polynomially ambiguous:**

Every word $w$ has at most $\mathcal{O}(|w|^k)$ accepting run.




Polynomially ambiguous
iff
unambiguous on SCCs

Domain: $a^+(a^+b^+)^+$

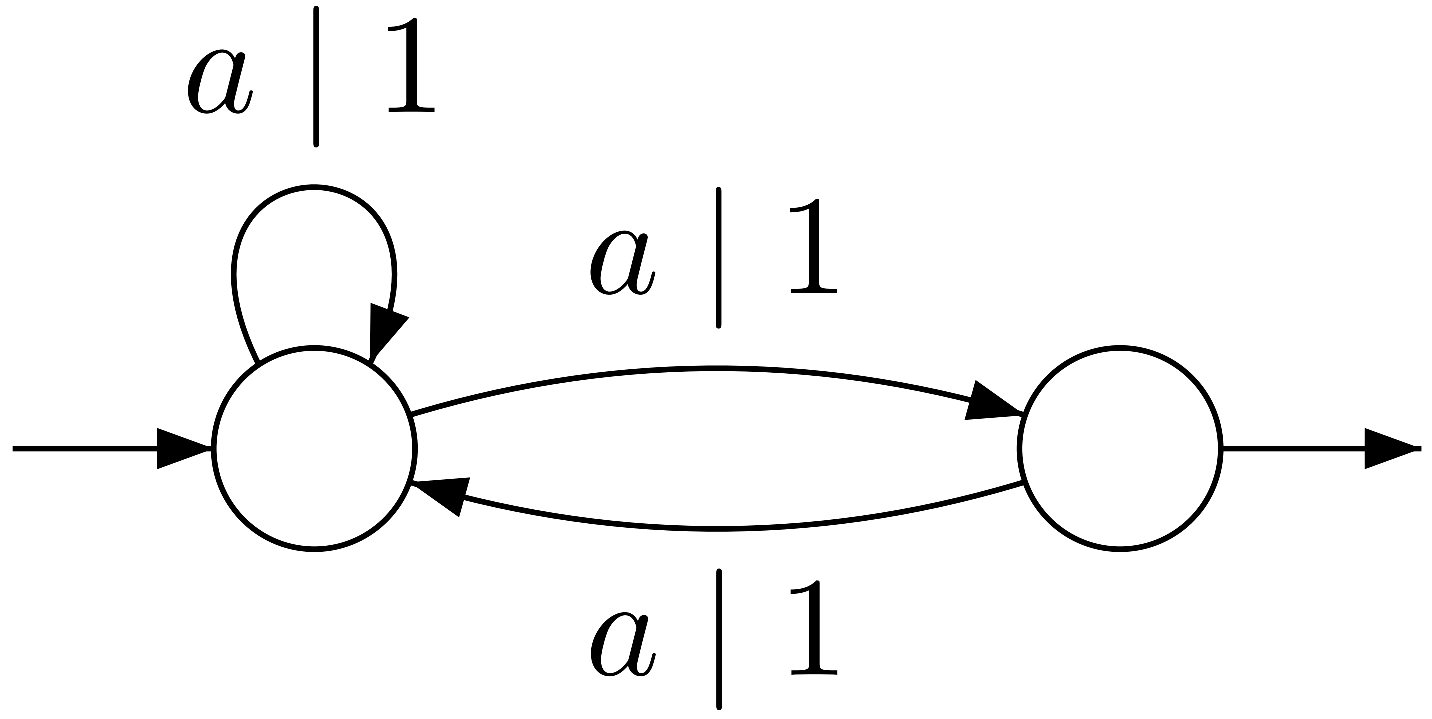
Quadratic ambiguity

Input word: $w \in a^n a(b^+a^+)^* b^p$

- $n \times p$ accepting runs on $w$
- Less than $|w|^2$ accepting runs on $w$

# Ambiguity in automata

General case: Exponential ambiguity




$\mathscr{A}$ is aperiodic: index 2

Domain: $a^+$

$F_n$: number of accepting runs on $a^n$

$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$

Fibonacci numbers: exponential ambiguity

# Aperiodic weighted Automata

Manfred Droste, P.G., 2019

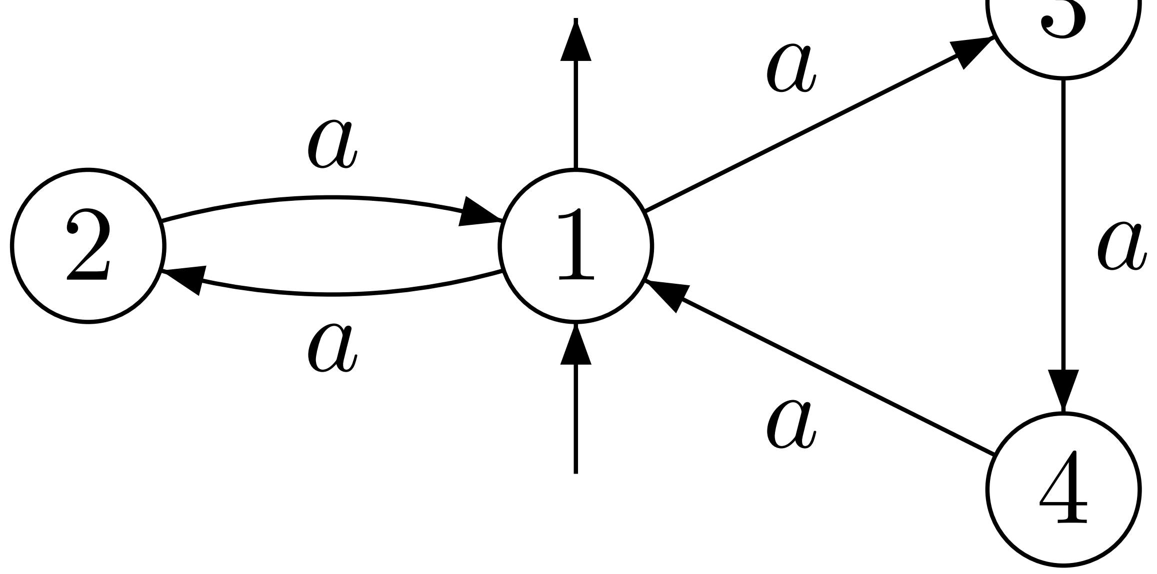
Strict

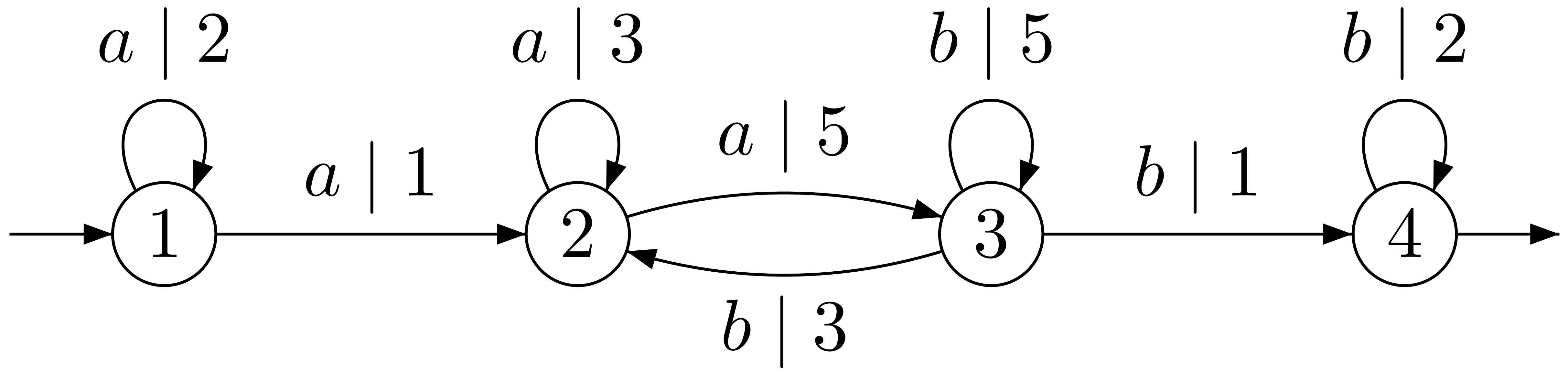
| | | |
|---|---|---|
| Exponentially ambiguous | | |
| Polynomially ambiguous | $\Longleftrightarrow$ | wFO |
| Finitely ambiguous | $\Longleftrightarrow$ | wFO without $\sum_x$ |
| Unambiguous | $\Longleftrightarrow$ | wFO without $\sum_x$ and without + |

# Aperiodic vs Counter-free

Aperiodic: $\exists n \geq 1, \forall u \in \Sigma^+, \forall p, q \in Q, \quad p \xrightarrow{u^n} q \iff p \xrightarrow{u^{n+1}} q$

Counter-free: $\forall k \geq 1, \forall u \in \Sigma^+, \forall p \in Q, \quad p \xrightarrow{u^k} p \implies p \xrightarrow{u} p$

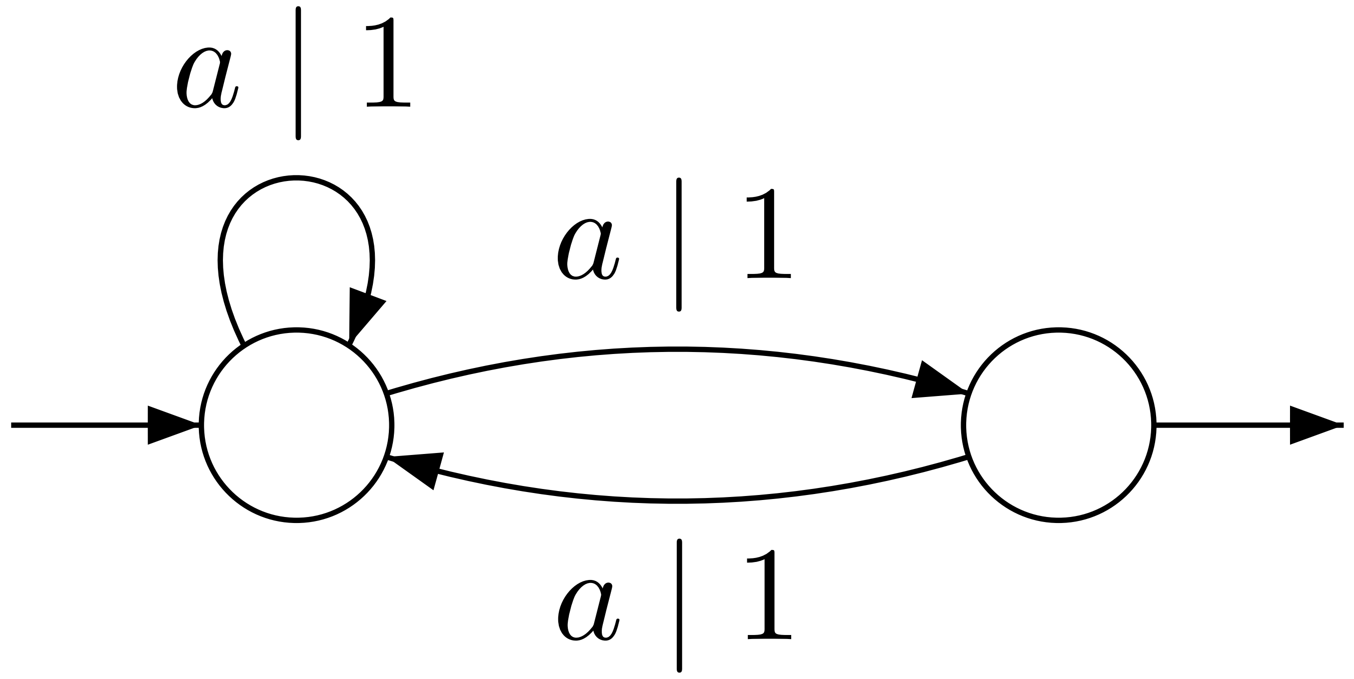




Counter-free $\implies$ Aperiodic

Aperiodic $\implies$ Counter-free if $\mathscr{A}$ is polynomially ambiguous (SCC-unambiguous)

# Poly-ambiguous vs Exp-ambiguous




Fibonacci numbers: $[\![\mathscr{A}]\!](a^n) = F_n$

Assume $[\![\mathscr{B}]\!](a^n) = F_n$ for some aperiodic polynomially ambiguous wA.

Polynomially ambiguous & Aperiodic $\Longrightarrow$ SCC-unambiguous & counter-free

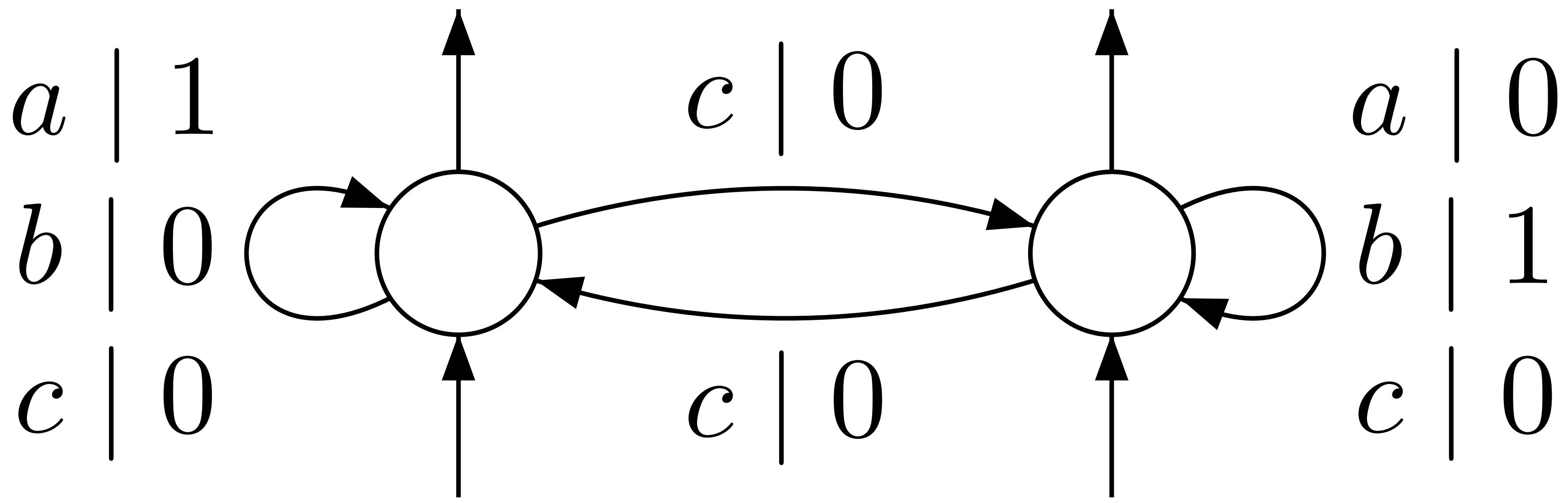
SCC-unambiguous & counter-free & $\Sigma = \{a\} \Longrightarrow$ SCCs are just self-loops

$\mathbb{N}_{+,\times} = (\mathbb{N}, +, \times, 0, 1)$ semiring & $F_n = o(2^n) \Longrightarrow$ weight(self-loop) $\leq 1$

Then, wt$(\rho) \leq K$ for some constant $K \in \mathbb{N}$

Therefore, $[\![\mathscr{B}]\!](a^n) \leq K \times$ poly$(n) = o(F_n)$, a contradiction.

# Poly-ambiguous vs Exp-ambiguous




$\mathbb{N}_{\max,+} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ : max-plus semiring.

$\mathscr{A}$ is exponentially ambiguous & counter-free.

Let $w = w_0 c w_1 c \cdots c w_n$ with $w_i \in \{a, b\}^*$. Then,

$$[\![\mathscr{A}]\!](w) = \sum_{i=0}^{n} \max\{\,|w_i|_a, |w_i|_b\,\}.$$

This function cannot be computed with an aperiodic poly-ambiguous wA.

# Aperiodic weighted Automata

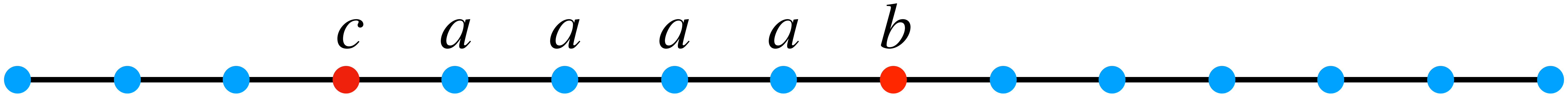
Manfred Droste, P.G., 2019

Strict ↑

| | | |
|---|---|---|
| Exponentially ambiguous | | |
| Polynomially ambiguous | $\iff$ | wFO |
| Finitely ambiguous | $\iff$ | wFO without $\sum_x$ |
| Unambiguous | $\iff$ | wFO without $\sum_x$ and without + |

# Linear Temporal Logic (LTL)

$$\varphi ::= \top \mid a \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{X}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi \mid \mathsf{Y}\,\varphi \mid \varphi\,\mathsf{S}\,\varphi$$



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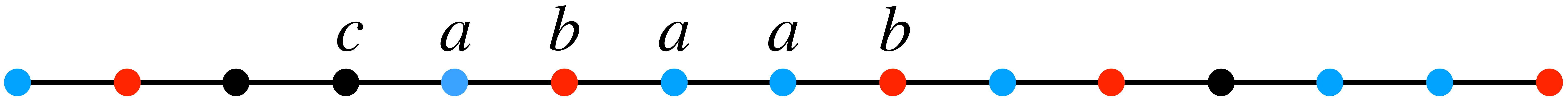



Example: $L = (ab + aab + c)^+$ is LTL-definable

- $\mathsf{F}\,\varphi = \top\,\mathsf{U}\,\varphi$ and $\mathsf{G}\,\varphi = \neg\mathsf{F}\neg\varphi$

Theorem [Kamp, PhD 1968]

FO = LTL.

# Weighted Linear Temporal Logic (wLTL)

LTL $\varphi ::= \top \mid a \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{X}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi \mid \mathsf{Y}\,\varphi \mid \varphi\,\mathsf{S}\,\varphi$

step-wLTL $\alpha ::= r \mid \varphi\,?\,\alpha : \alpha$

wLTL $\Phi ::= \mathbf{0} \mid \mathsf{L}\,\alpha \mid \langle\alpha\rangle\Phi \mid \alpha\,\mathsf{U}\,\Phi \mid \Phi + \Phi \mid \varphi\,?\,\Phi : \Phi$

Semantics: $w \in \Sigma^+$ and $i \in \mathrm{pos}(w)$

$$[\![r]\!]_i^w = r$$

$$[\![\varphi\,?\,\alpha_1 : \alpha_2]\!]_i^w = \begin{cases} [\![\alpha_1]\!]_i^w & \text{if } w, i \models \varphi \\ [\![\alpha_2]\!]_i^w & \text{otherwise.} \end{cases}$$

$$[\![\mathsf{L}\,\alpha]\!]_i^w = \begin{cases} [\![\alpha]\!]_i^w & \text{if } i = |w| \\ 0 & \text{otherwise.} \end{cases}$$

$$[\![\langle\alpha\rangle\Phi]\!]_i^w = \begin{cases} [\![\alpha]\!]_i^w \cdot [\![\Phi]\!]_{i+1}^w & \text{if } i < |w| \\ 0 & \text{otherwise.} \end{cases}$$

$$[\![\alpha\,\mathsf{U}\,\Phi]\!]_i^w = \sum\nolimits_{i \le k \le |w|} \left( \prod\nolimits_{i \le j < k} [\![\alpha]\!]_j^w \right) \times [\![\Phi]\!]_k^w$$

# Weighted Linear Temporal Logic (wLTL)

LTL $\varphi ::= \top \mid a \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{X}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi \mid \mathsf{Y}\,\varphi \mid \varphi\,\mathsf{S}\,\varphi$

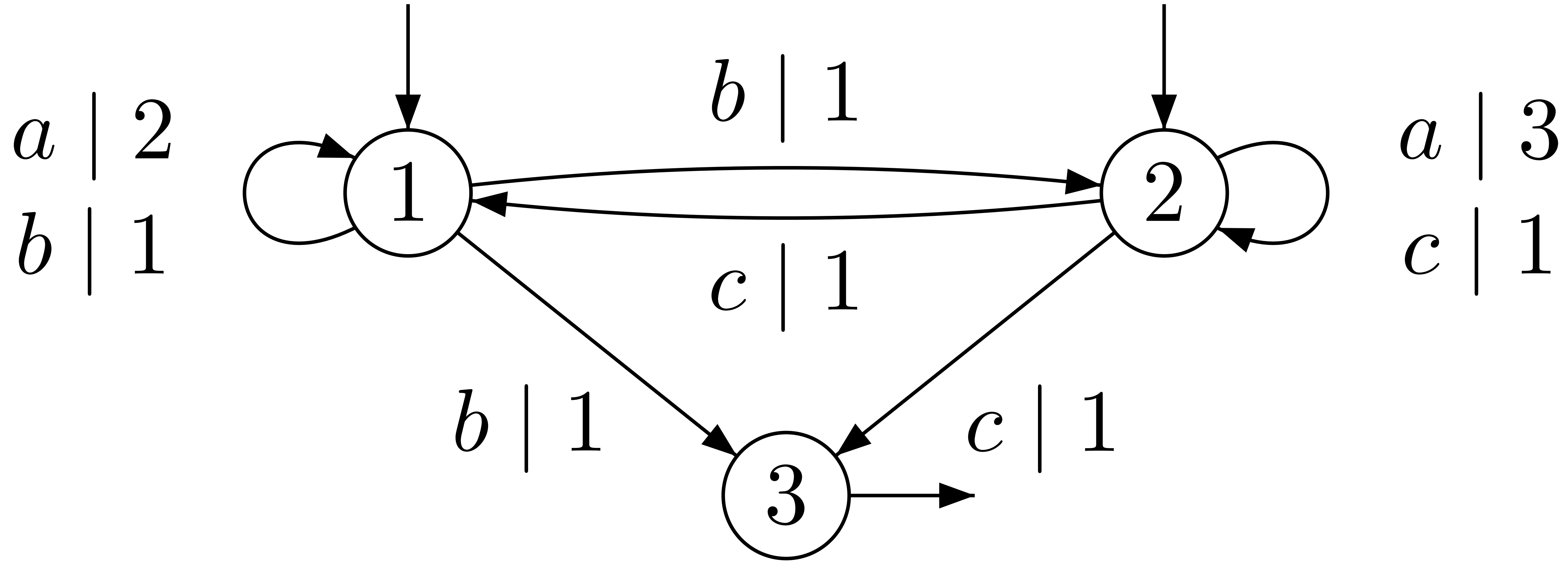
step-wLTL $\alpha ::= r \mid \varphi\,?\,\alpha : \alpha$

wLTL $\Phi ::= \mathbf{0} \mid \mathsf{L}\,\alpha \mid \langle\alpha\rangle\Phi \mid \alpha\,\mathsf{U}\,\Phi \mid \Phi + \Phi \mid \varphi\,?\,\Phi : \Phi$

Example: $L = (ab + aab + c)^+$ is LTL-definable

- $\varphi_L = \mathsf{G}\big((b \to \mathsf{Y}\,a) \wedge (a \to (\mathsf{X}\,b \vee \mathsf{X}(a \wedge \mathsf{X}\,b)))\big)$
- $\alpha = (a \vee c)\,?\,1 : \big((\mathsf{Y}\,\mathsf{Y}\,a)\,?\,3 : 2\big)$
- $\Phi = \varphi_L\,?\,\mathsf{G}\,\alpha : \mathbf{0}$ $\qquad\mathsf{G}\,\alpha = \alpha\,\mathsf{U}\,\mathsf{L}\,\alpha$
- For $w \in L$, let $n = |w|_{ab}$ and $m = |w|_{aab}$. Then, $[\![\Phi]\!]_1^w = 3^m \times 2^{n-m}$

# Unambiguous aperiodic wA vs wLTL




Domain of $\mathscr{A}$: $L = (a^*b + a^*c)^+ = \Sigma^*(b+c)$

- $\varphi_L = \mathsf{G}\big(b \vee c \vee \mathsf{X}\,\top\big)$
- $\alpha = (b \vee c)\,?\,1 : \big((a \,\mathsf{U}\, b)\,?\,2 : 3\big)$
- $\Phi = \varphi_L\,?\,\mathsf{G}\,\alpha : \mathbf{0}$
- Then, $[\![\Phi]\!]_1^w = [\![\mathscr{A}]\!](w)$ for all $w \in \Sigma^+$

# wLTL(G) without + and U

LTL $\varphi ::= \top \mid a \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{X}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi \mid \mathsf{Y}\,\varphi \mid \varphi\,\mathsf{S}\,\varphi$

step-wLTL $\alpha ::= r \mid \varphi\,?\,\alpha : \alpha$

wLTL $\Phi ::= \mathbf{0} \mid \mathsf{G}\,\alpha \mid \langle\alpha\rangle\Phi \mid \varphi\,?\,\Phi : \Phi$

$\mathsf{G}\,\alpha = \alpha\,\mathsf{U}\,\mathsf{L}\,\alpha$

Semantics: $w \in \Sigma^+$ and $i \in \mathrm{pos}(w)$ $\quad [\![\mathsf{G}\,\alpha]\!]_i^w = \prod_{i \leq j \leq |w|} [\![\alpha]\!]_j^w$

wLTL(G) without + and U is contained in wFO without binary (+) or first-order $\left(\sum_x\right)$ sums.

Theorem [Kamp, PhD 1968]

FO = LTL.

For each $\alpha$ in step-wLTL there is an equivalent $\alpha'(z)$ in step wFO: $[\![\alpha]\!]_i^w = [\![\alpha']\!]_{[z\mapsto i]}^w$

We get $[\![\mathsf{G}\,\alpha]\!]_1^w = \prod_{1 \leq i \leq |w|} [\![\alpha]\!]_i^w = \prod_{1 \leq i \leq |w|} [\![\alpha']\!]_{[z\mapsto i]}^w = [\![\prod_z \alpha']\!]^w$

# Unambiguous aperiodic wA vs wLTL

Let $\mathscr{A}$ be an aperiodic and unambiguous wA

- $L = \mathrm{dom}(\mathscr{A})$ is LTL-definable by some sentence $\varphi_L$ in LTL
- For states $p, q$ we construct $\psi_p \in \mathsf{LTL}(\mathsf{Y}, \mathsf{S})$ and $\phi_q \in \mathsf{LTL}(\mathsf{X}, \mathsf{U})$ such that

$$I \xrightarrow{u} p \quad \text{iff} \quad u, |u| \models \psi_p \qquad \text{and} \qquad q \xrightarrow{v} F \quad \text{iff} \quad v, 1 \models \phi_q$$

- For each transition $\delta = p \xrightarrow{a} q$, let $\varphi_\delta = \mathsf{Y}\, \psi_p \wedge a \wedge \mathsf{X}\, \phi_q$.

  For all $w \in L$ and $i \in \mathrm{pos}(w)$

  $w, i \models \varphi_\delta$ iff $\delta$ is the $i$-th transition in the accepting run of $\mathscr{A}$ on $w$

- Construct $\alpha$ in step-wLTL such that $[\![\alpha]\!]_i^w = \mathrm{wt}(\delta)$ when $w, i \models \varphi_\delta$
- Let $\Phi = \varphi_L \,?\, \mathsf{G}\, \alpha : \mathbf{0}$. Then, $[\![\Phi]\!]_1^w = [\![\mathscr{A}]\!](w)$ for all $w \in \Sigma^+$

# Unambiguous aperiodic wA vs wLTL

Theorem: The following classes are expressively equivalent:

1. Aperiodic unambiguous wA
2. wFO sentences without binary (+) or first-order $\left(\sum_x\right)$ sums.
3. wLTL(G) sentences without binary (+) sums or weighted until.

We just saw $(1) \implies (3)$ and $(3) \implies (2)$.

$(2) \implies (1)$ proved in Droste-Gastin 2019.

# Aperiodic wA vs wFO vs wLTL

| | | | | |
|---|---|---|---|---|
| Exponentially ambiguous | | | | |
| Polynomially ambiguous | $\Longleftrightarrow$ | wFO | $\overset{?}{\Longleftrightarrow}$ | wLTL |
| Finitely ambiguous | $\Longleftrightarrow$ | wFO without $\sum_x$ | $\Longleftrightarrow$ | wLTL without U |
| Unambiguous | $\Longleftrightarrow$ | wFO without + and $\sum_x$ | $\Longleftrightarrow$ | wLTL no + and U |

# wFO vs wLTL

Example: Let $L \subseteq \Sigma^+$ be a language defined by the FO sentence $\varphi$.

$$\Phi_{\mathsf{pref}} = \sum_x \varphi^{\leq x} \,?\, \mathbf{1} : \mathbf{0}$$

Number of prefixes in $L$

where $\mathbf{1} = \prod_z 1$ and $\varphi^{\leq x}$ is $\varphi$ relativized to the prefix $[1,x]$

Example: Let $L \subseteq \Sigma^+$ be a language defined by the formula $\psi \in \mathsf{LTL}(\mathsf{Y}, \mathsf{S})$.

$$\Phi = \big(1 \;\mathsf{U}\; \langle \psi \,?\, 1 : \mathbf{0} \rangle \mathsf{G}\, 1\big) + \big(1 \;\mathsf{U}\; \mathsf{L}(\psi \,?\, 1 : \mathbf{0})\big)$$

# wFO vs wLTL

Example: Let $L \subseteq \Sigma^+$ be a language defined by the FO sentence $\varphi$.

$$\Phi_{\text{inf}} = \sum_{x,y} (x \leq y \wedge \varphi^{[x,y]}) \mathbin{?} \mathbf{1} : \mathbf{0}$$

where $\mathbf{1} = \prod_z 1$ and $\varphi^{[x,y]}$ is $\varphi$ relativized to the infix $[x, y]$

Number of factors in $L$

# Thank you for your attention!